AERODYNAMIC OPTIMIZATION OF A COAXIAL PROPROTOR

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Abstract

Fundamental aerodynamic performance issues that are associated with the design of a contrarotating, coaxial proprotor are discussed. The simple (global) momentum theory of a coaxial rotor was derived for both thrust and torque balances. This theory was validated against thrust and power measurements for a coaxial rotor system, with good agreement. The blade element momentum theory (BEMT) for a coaxial rotor was formally derived to solve for the distribution of local airloads over the upper and lower rotors. The BEMT was found to agree well with measured coaxial rotor performance, and gives better results when compared to experiments than the simple momentum theory alone. The results from the BEMT were further validated using a free-vortex wake analysis of the coaxial rotor, also with good agreement. A new figure of merit expression for a coaxial rotor was established based on a thrust sharing ratio between the upper and lower rotors, reflecting the fact that each rotor operates at a different disk loading at the torque balanced condition. It is shown that the optimum case for a maximum net figure of merit occurs when both the upper and lower rotors operate with uniform disk loadings at balanced torques. The ideal case also corresponds to constant and uniform inflow on the upper rotor but a double-valued uniform inflow on the lower rotor (one inside the wake boundary from the upper rotor and one outside). The BEMT was used to design an optimum coaxial rotor for minimum losses and a maximum figure of merit. The optimum twist for the lower rotor corresponds to a double hyperbolic twist, with the break point being at the location where the wake boundary from the upper rotor impinges the lower rotor. Optimization of the blade planform is required to produce distributions of lift coefficients that minimize profile losses by having all the blade sections work close to their best lift-to-drag ratios, which is shown to require different tapers and solidities on the upper and lower rotors. Finally, a composite propulsive efficiency for a coaxial proprotor has been defined, again recognizing the unequal rotor load sharing in the torque balanced condition. Results that were obtained for an optimum hybrid proprotor design combining good propulsive efficiency and high hovering figure of merit are discussed.

Nomenclature

\begin{align*}
A & \quad \text{Rotor disk area (for one rotor)} \\
A_c & \quad \text{Contracted slipstream wake area} \\
c & \quad \text{Blade chord} \\
Cd & \quad \text{Drag coefficient} \\
Cd_l & \quad \text{Zero lift (viscous) drag coefficient} \\
C_l & \quad \text{Lift coefficient} \\
C_{l_{c}} & \quad \text{Lift curve slope} \\
C_{T} & \quad \text{Rotor thrust coefficient, } T / \rho A \Omega^2 R^2 \\
C_{P} & \quad \text{Rotor power coefficient, } P / \rho A \Omega^3 R^3 \\
\zeta & \quad \text{Wake (vortex) age} \\
V & \quad \text{Vehicle weight} \\
\lambda & \quad \text{Non-dimensional induced velocity, } v/\Omega R \\
\lambda_d & \quad \text{Tip speed ratio, } V_{\infty}/\Omega R \\
\mu & \quad \text{Advance ratio} \\
\rho & \quad \text{Flow density} \\
\sigma & \quad \text{Rotor solidity, } N_{b} c / \pi R \\
\psi & \quad \text{Azimuth angle} \\
\psi_b & \quad \text{Blade azimuthal location} \\
\Omega & \quad \text{Rotational speed of the rotor}
\end{align*}

Introduction

Coaxial rotors are not a new idea, the concept having been used since the dawn of experiments with helicopters in the nineteenth century. The earliest coaxial rotor designs stem from patented ideas of Bright in 1861 and the models of d’Amécourt in 1862,
through to the test stand made by Igor Sikorsky in 1910, to the
human-carrying helicopter prototypes of Emile Berliner, Cor-
radino d’Ascanio, and Louis Breguet in the 1930s. Contemporary
sources (Refs. 1, 2) suggest that at least 35 prototype heli-
copters that used coaxial rotors had been built (but not necessarily
flown successfully) prior to 1945.

The preference for symmetric rotor configurations such as
c coaxials, lateral side-by-side rotors, and tandems was reversed by the
success of Sikorsky’s single rotor configuration in the early 1940s. However, the post-World War II era still saw several
more prototype helicopters appear with coaxial rotors, many
of which flew successfully, including those from Hiller, Bendix,
Gyrodyne, Breguet, Kamov, and others. In the 1970s, a coaxial
rotor was used by Sikorsky for their Advancing Blade Concept
(ABC) demonstrator (Ref. 3), although the aircraft never went
into production. Only the Kamov company from Russia has
been successful in putting the coaxial rotor configuration into
production, starting with the Ka-6/8 helicopters in the late 1940s
to the Ka-50 in the 1990s.

In 2005, Sikorsky again proposed the use of a coaxial rotor
system, this time for a variety of helicopters, including heavy-
light transporters. In 2002, the Baldwin Technology Company
proposed using a coaxial rotor for the Mono Tiltrotor (MTR).
The underpinning of the MTR concept is a single, contrarotating,
coaxial rotor system, which must have the hovering efficiency
of a helicopter but morphs to wing-borne flight for cruise with
the rotor tilted forward and acting as a propulsor. The ability to
design an optimum coaxial rotor for efficient static thrust perfor-
mance while retaining good propulsive efficiency as a propulsor
is essential to the success of the MTR concept (Refs. 4, 5). The
ongoing development of the MTR has provided the primary mo-
tivation for the work contained in the present paper.

Coleman (Ref. 6) gives a good summary of coaxial rotor
designs along with a comprehensive list of relevant citations
on performance, wake characteristics, and proposed methods
of performance analysis. See also Andrew (Ref. 7), Saito &
Azuma (Ref. 8), and Zimmer (Ref. 9) for further details on coax-
ial rotor performance. One most often-cited advantage of the
contrarotating coaxial rotor design as applied to a helicopter is
that the overall rotor diameter can be reduced (for a given ve-
cicle gross weight) because the thrust vector from each rotor is
directed vertically upward, and so each rotor provides a maxi-
mum contribution to vertical thrust to overcome vehicle weight.
In addition, no tail rotor is required, so that all engine power can
be devoted to providing useful lifting performance and that no significant power is wasted for anti-torque and directional con-
trol.

However, on a coaxial rotor the two rotors and their wakes in-
teract with one another, hence producing a somewhat more com-
licated flow field than is found with a single rotor. This interact-
ing flow generally incurs a loss of net rotor system aerodynamic
efficiency, although there have been claims to the contrary.
Highly loaded propellers, for example, sometimes see gains in
propulsive efficiency when operated in “tandem” (i.e., coaxially,
one behind the other) because they recover swirl momentum
losses in the wake – see Eiffel (Ref. 10), Lesley (Ref. 11),
and Weick (Ref. 12). This advantage, however, is mostly lost
with the low disk loadings typical of helicopter rotors.

Taking into account both aerodynamic and mechanical losses
suggests that as a hovering aircraft platform, a coaxial rotor
system is just as efficient as a conventional (single main rotor
tail rotor) configuration. However, the typically higher pro-
file and parasitic drag (from the exposed mast and controls) of

Momentum Theory for a Coaxial

The performance of a coaxial rotor system can first be exam-
ined by means of the simple (or global) one-dimensional mo-
momentum theory, i.e., the Glauert theory (Ref. 13). See also
Johnson (Ref. 14). This theory uses the fluid mass, momentum,
and energy conservation equations in an integral form. The con-
servation laws are applied to a control volume starting above
and ending below the rotors and encompassing the limits of the
two rotor disks. The problem can be tackled completely analyti-
cally without recourse to a numerical solution, which is a signif-
icant advantage. The specification a priori of an operating state
(i.e., the total rotor system thrust) allows the induced velocity
and power requirements of each rotor and of the system to be
derived. There are no viscous or compressibility losses allowed
for in the simple momentum theory approach – all losses are
induced in nature and so the results represent the minimum the-
oretical source of losses for the dual rotor system based on the
stated assumptions. It is for this reason that in helicopter anal-
ysis the simple momentum theory is used as a datum to define the
operating efficiency of a rotor system, i.e., to define its figure of
merit.

There are four momentum theory cases of interest for a coax-
ial rotor system: Case 1: The two rotors rotate in the same plane
(or very nearly so in practice) and are operated at the same thrust;
Case 2: The two rotors rotate in the same plane but are operated
at a balanced (equal and opposite) torque; Case 3: The rotors
are operated at the same thrust but the lower rotor operates in
the fully developed slipstream (i.e., the vena contracta) of the
upper rotor; Case 4: The rotors are operated at balanced torque
with the lower rotor operating in the vena contracta of the upper
rotor.

In the first case, assume that the two rotor planes of rotation
are sufficiently close together (Fig. 1) and that each rotor pro-
vides an equal fraction of the total system thrust (2T) where
T0 = T1 = T = W/2. The basis on which to compare the coaxial
A corollary to this result is that for coaxial rotors that rotate in the same plane, the induced power factor is independent of the thrust sharing between the rotors, i.e., when $T_u \neq T_l$ (see appendix).

In practice, the two rotors of coaxial rotor system are never operated at the same thrust but instead at whatever individual thrust levels are necessary to give a balanced (equal and opposite) torque on the two rotors as a system (Case 2). However, in the case where the two rotors are sufficiently close that they rotate in substantially the same plane at the same thrust, then they must also require the same torque (power). This is because both of the rotors share the same value of induced velocity (Fig. 2). This means that for a coaxial rotor system with the rotors in the same plane operated at either the same thrust and/or the same torque then $\kappa_{\text{int}} = 1.414$. Notice that in the case where the rotors operate in the same plane then a torque balance can be achieved only if $T_u = T_l$; other variations of thrust sharing will spoil the torque balance.

The forgoing simple momentum analysis of the coaxial rotor problem has been shown to be overly pessimistic when compared with experimental measurements for closely spaced coaxial rotors – see Harrington (Ref. 18) and Dingeldein (Ref. 19) and the various results given in the review by Coleman (Ref. 6). One often cited reason for the overprediction of induced power is related to the actual (finite) spacing between the two rotors. Generally, on practical coaxial designs the rotors are spaced sufficiently far apart to prevent inter-rotor blade collisions that the lower rotor always operates in the $\text{vena contracta}$ of the upper rotor. This is justified from the flow visualization results of Taylor (Ref. 20), for example, where the wake of the upper rotor contracts quickly (within 0.25R below the rotor) so it can be considered fully contracted by the time it is ingested by the lower rotor. The ideal wake contraction ratio is 0.707, but in practice it is found closer to 0.8. If it is assumed the lower rotor does not affect the wake contraction of the upper rotor, then based on ideal flow considerations one-half of the disk area of the lower rotor must operate in the slipstream velocity induced by the upper rotor. This is a more difficult physical problem to model, in general, because it involves wake–blade interactions and local viscous effects, but it can initially be approached by a similar method of analysis using the principles of the simple momentum theory.

The flow model for this case is shown in Fig. 2. In the first instance, assume that the two rotors operate at the same thrust, $T_u = T_l$ (Case 3). The induced power factor from interference, $\kappa_{\text{int}}$, is given by (see appendix for full details)

$$\kappa_{\text{int}} = \frac{(P_i)_{\text{coax}}}{(2P_i)_{\text{isolated}}} = \frac{2.562 TV_h}{2Tv_h} = 1.281$$

which is a 28% increase in induced losses compared to a 41% increase when the two rotors have no vertical separation. This is closer to the values that can be indirectly deduced from most experiments with coaxial rotor systems – see, for example, Dingeldein (Ref. 19).

A comparison of performance on the basis of equal balanced torque (Case 4) between the upper and lower rotor is, however, a much more realistic operational assumption for a coaxial rotor. For this condition (see appendix) the induced power factor $\kappa_{\text{int}}$ is reduced to

$$\kappa_{\text{int}} = \frac{2.4375V_h}{2V_h} = 1.219$$

which now represents a 22% increase compared to the case when the two rotors are operated in isolation, and this is just slightly less than the 28% loss when the rotors are operated at equal thrusts. A summary of the simple momentum theory results for
the coaxial rotor system is shown in Table 1. Notice that these results represent the minimum induced losses for each coaxial configuration and so set the datum for comparison with any real coaxial rotor system in terms of an operating figure of merit, etc.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Induced power factor, $\kappa_{int}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.414</td>
</tr>
<tr>
<td>2</td>
<td>1.414</td>
</tr>
<tr>
<td>3</td>
<td>1.281</td>
</tr>
<tr>
<td>4</td>
<td>1.219</td>
</tr>
</tbody>
</table>

Table 1: Summary of minimum induced power factors for coaxial rotors using the simple momentum theory.

BEMT for a Coaxial Rotor

The blade element momentum theory (BEMT) is a hybrid rotor analysis method that was developed in analytical form for propeller analysis (Ref. 12). This approach is the essence of Froude’s original differential theory for single propellers in axial motion (Ref. 21). A variation of the technique was first proposed for use with single helicopter rotors by Gessow (Ref. 22) and Gessow & Myers (Ref. 23). The BEMT has since become a standard method of preliminary rotor analysis, although apparently it has not been previously set up and solved for a coaxial rotor system.

The BEMT combines the basic principles from both the blade element (i.e., section-by-section) and momentum approaches. The principles involve the invocation of the equivalence between the circulation and momentum theories of lift, and apply the conservation laws to an annulus of the rotor disk. The incremental thrust on this annulus may be calculated using a differential form of the momentum theory with the assumption that successive rotor annuli have no mutual effects on each other, i.e., a 2-D assumption. The BEMT integrates to the simple momentum theory in the limiting case of uniform disk loading (or uniform inflow for a single rotor).

The BEMT flow model for the coaxial rotor is shown in Fig. 3. It is assumed that the lower rotor operates partly in the fully developed slipstream of the upper rotor, which in practice, as previously mentioned, is probably the more correct assumption for a coaxial rotor system. The BEMT equations can be formalized for the upper and lower rotors, again by invoking the previously made assumption that the upper rotor affects the flow into the lower rotor but the lower rotor does not affect the flow at the upper rotor. The theory can also be generalized in terms of an axial climb velocity, say $V_c = V_\infty$. This is useful because the results allow an optimum blade design solution to be determined for the case where the rotors also act as a propulsive system in forward flight.

Consider first the upper rotor. The incremental thrust on the rotor annulus is the product of the mass flow rate through the annulus and twice the induced velocity at that section. In this case, the mass flow rate over the annulus is

$$d\dot{m} = \rho dA(V_\infty + v_\lambda) = 2\pi\rho(V_\infty + v_\lambda)v_\lambda dy$$

(4)

so that the incremental thrust on the annulus is

$$dT_u = 2\rho(V_\infty + v_\lambda)v_\lambda dA = 4\pi\rho(V_\infty + v_\lambda)v_\lambda y dy$$

(5)

The blade element momentum theory (BEMT) is a hybrid rotor system.
and $\phi$ is the induced inflow angle ($= \lambda(r)/r$). In application the function $F$ can also be interpreted as a reduction factor applied to the change in fluid velocity as it passes through the control volume, i.e., by using an equation of the form

$$dC_T = 4F\lambda^2 r \, dr$$  \hspace{1cm} (11)

Now, using the blade element theory (i.e., the circulation theory of lift) the incremental thrust produced on the same annulus of the disk is

$$dC_T = \frac{1}{2} \sigma C_l r^2 \, dr = \frac{\sigma C_{l_\infty}}{2} \left( \theta_css - \lambda \right) \, dr$$  \hspace{1cm} (12)

where $\theta_cs$ is the blade pitch distribution on the upper rotor. Therefore, equating the incremental thrust coefficients from the momentum and blade element theories (using Eqs. 7 and 12) gives

$$\sigma C_{l_\infty} \left( \theta_css - \lambda \right) = 4F\lambda(\lambda - \lambda_\infty) \, r$$  \hspace{1cm} (13)

or rearranging in terms of $\lambda$ gives

$$\lambda^2 + \left( \frac{\sigma C_{l_\infty}}{8F} - \lambda_\infty \right) \lambda - \frac{\sigma C_{l_\infty}}{8F} \theta_css = 0$$  \hspace{1cm} (14)

This quadratic equation in $\lambda$ has the solution

$$\lambda(r, \lambda_\infty) = \sqrt{\frac{\sigma C_{l_\infty}}{16F} - \frac{\lambda_\infty}{2} + \frac{\sigma C_{l_\infty}}{8F} \theta_css} - \frac{\sigma C_{l_\infty}}{8F} \lambda_\infty = \frac{\sigma C_{l_\infty}}{16F} \left( \frac{\lambda_\infty}{2} + \frac{\sigma C_{l_\infty}}{8F} \theta_css \right) - \frac{\sigma C_{l_\infty}}{8F} \lambda_\infty$$  \hspace{1cm} (15)

Equation 15 is solved numerically over a series of discretized elements distributed radially over the rotor blade (disk). Because $F$ is a function of the inflow $\lambda$, this equation cannot be solved immediately because $\lambda$ is initially unknown. Therefore, it is solved iteratively by first calculating $\lambda$ using $F = 1$ (corresponding to $N_b \rightarrow \infty$) and then finding $F$ from Eq. 9 and recalculating $\lambda$ from the numerical solution to Eq. 15. Convergence is rapid and is obtained in at most five iterations.

From the inflow solution, the corresponding distributions of thrust and power are easily determined, and the total thrust and power are found by integrating the equations

$$C_T = \int_{r=0}^{r=1} \lambda^2 r \, dr$$  \hspace{1cm} (16)

and

$$C_p = \int_{r=0}^{r=1} \lambda_\infty \, dC_T = \int_{r=0}^{r=1} \lambda_\infty \lambda^2 r \, dr$$  \hspace{1cm} (17)

which are both evaluated numerically element-by-element using the discretized solution of the inflow.

The same principles can be applied to the lower rotor. However, in this case the lower rotor operates in the vena contracta of the upper rotor. The slipstream velocity in the streamtube of the upper rotor can be defined on the basis of the assumed radial contraction of the wake, i.e., the contracted wake area $A_c = \pi r_c^2$. In the ideal case, $r_c = 2^{-1/2} = 0.707$ or $A/A_c = 2$. In this case, the inner area of the lower rotor encounters incoming streamtubes with velocity $V_w + 2V_u$ in the ideal case, or $V_w + (A/A_c) V_u$ in the general case where the wake contraction from the upper rotor is otherwise specified. Following the same steps as for the upper rotor, for points on the disk within the slipstream area from the upper rotor (i.e., for $r \leq r_c$) the inflow distribution is given by solving

$$\lambda(r, \lambda_\infty) = \sqrt{\frac{\sigma C_{l_\infty}}{16F} - \frac{\lambda_\infty}{2} + \frac{\sigma C_{l_\infty}}{8F} \theta_f r} - \left( \frac{\sigma C_{l_\infty}}{16F} - \frac{\lambda_\infty}{2} \right)$$  \hspace{1cm} (18)

Equations 15, 18 and 19, therefore, allow for a solution of the inflow at discrete radial positions over the upper and lower rotors for any given blade pitch, blade twist distribution, planform (i.e., chord distribution) and airfoil section (i.e., implicitly through the effect of lift-curve-slope and zero-lift angle of attack). When the inflow is obtained, the rotor thrust and induced power may then be found by numerical integration across the rotor disk using Eqs. 16 and 17 for the upper rotor, and with an equivalent set of equations for the lower rotor.

The BEMT approach generally has good validity for airloads prediction except near the blade tips and, in the case of a coaxial rotor, also in the region where the upper wake boundary impinges on the lower rotor. The removal of this 2-D restriction requires a better treatment of the problem using vortex theory (see next section). However, a good approximation to the tip-loss effect on the inflow distribution can be made using Prandtl’s circulation-loss function. There is no equivalent loss function, however, to represent the extra local induced losses where the wake boundary from the upper rotor interacts with the lower rotor, so the BEMT would generally be expected to underpredict the induced losses here.

A primary outcome of formulating a BEMT analysis for a coaxial rotor system is that it allows the blade twist and planform distributions on the upper and lower rotors to be designed to give a specified performance, including operating the system for minimum induced and profile losses. This approach, therefore, can be used to seek an optimum rotor design for a given operating state, but only if the BEMT can be suitably justified against experiments or more advanced forms of aerodynamic analysis of the coaxial rotor system.

**FVM Analysis of a Coaxial**

The FVM offers a higher-level aerodynamic analysis of the coaxial rotor problem, albeit at considerably higher computational cost. The problem of the coaxial rotor using a FVM was examined by Bagai & Leishman (Ref. 15). The FVM also offers a further basis with which to validate the BEMT.

The FVM uses a Lagrangian description of vorticity in the flow field that is modeled using discrete line vortex filaments. The flow field is assumed to be incompressible, with all of the vorticity in the flow field being assumed to be concentrated within these vortex filaments. In a blade-fixed coordinate system, the governing equations for the wake are solved numerically by the application of finite-differences to approximate the space and time derivatives as they apply to the movement of
Lagrangian markers in the flow. In discretized form, the governing equation is

\[ \frac{\partial \mathbf{r}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{r} = D + \sum V = \nabla \mathbf{r} \]

(20)

where \( D \) is the temporal difference operator and \( D \) is the spatial difference operator, using temporal discretizations \( \Delta \) and space discretizations \( \Delta \), respectively. The displacements of the wake markers are found by integrating the equations using a second-order, time-accurate predictor-corrector scheme (PC2B scheme) developed by Bhagwat & Leishman (Refs. 16, 17). The induced velocities on the right-hand side of Eq. 20 are computed by the repeated application of the Biot–Savart law to the discretized vorticity field. This is the main computational expense that is associated with the FVM.

The coaxial rotor was modeled as rigid, articulated blades that execute independent, time-dependent flapping motion. Aerodynamically, each blade was modeled using a Weissinger-L lifting surface model. The solution to the bound circulation strengths on the blades were obtained by solving a system of equations that invoke the flow tangency condition as a summation of the influence of the bound and trailed circulations, in conjunction with the free stream velocity and any other external velocities. The wake strengths (circulation) are coupled to the gradient of circulation on the blades. The rotor airloads are closely coupled with the blade-flapping response, for which a coaxial rotor can be considered to have two independent responses coupled only through aerodynamics. After the blade lift and flapping response of both rotors has been simultaneously determined, the net aerodynamic forces and moments on the rotors were directly calculated by numerical integration of the blade element loads over the blade span and around the rotor azimuth.

The coaxial rotor must also be trimmed to meet specified equilibrium conditions (thrust and/or torque). Three main rotor control inputs were used; each rotor was numerically trimmed to remove cyclic flapping with respect to the shaft. The desired initial values of the control angles were obtained from vertical force equilibrium and by setting the rotor tip-path-plane (TPP) perpendicular to the hub axis, i.e., setting the cyclic flapping coefficients to zero. The requirements for torque balance \( \sum C_T \neq 0 \) at a net system thrust \( C_L \neq 0 \) were also imposed by finding a trim Jacobian and iterating the controls to the required thrust and/or torque balance condition.

**Results**

In the following subsections, results from the simple momentum theory, BEMT, and FVM analysis of the coaxial rotor system are described. The results include examples of validation with measured performance, where possible, and validation of the BEMT with the FVM.

**Validation of Simple Momentum Theory**

Despite the assumptions and relative simplicity of the moment-based theories for the coaxial rotor problem, their value becomes apparent when predictions are compared with measured rotor performance. Figures 4 and 5 shows the performance polar of single and coaxial rotors operating in hover. The measurements were taken from Harrington (Ref. 18) and are for two sets of rotors. Rotor system 1 has a solidity, \( \sigma \), of 0.027 with two blades per rotor (2\( \sigma \) = 0.054 when operated as a coaxial) and Rotor system 2 also has two blades per rotor with a solidity of 0.076 (2\( \sigma \) = 0.152 as a coaxial). Both sets of rotors have untwisted blades. The power for the single, two-bladed rotor was calculated from the simple momentum theory using

\[ P = \kappa \left( \frac{T^2}{2 \rho A} \right)^{3/2} + \rho A \left( \frac{\sigma C_d}{8} \right) \]

and for the coaxial rotor system using

\[ P = \kappa \int \left( \frac{2T^2}{2 \rho A} \right)^{3/2} + \rho A \left( \frac{2 \sigma C_d}{8} \right) \]

In this case, the assumed values \( \kappa = 1.1 \) and \( \kappa_{int} = 1.26 \) gives good agreement with the measurements (also assuming \( C_{d0} = 0.011 \)) and confirms that the coaxial rotor operates as two isolated rotors but with an interference effect accounted for by \( \kappa_{int} \). There is perhaps some evidence in these measurements of nonlinear aerodynamics at higher thrust coefficients, but the simple momentum theory shows what would be possible in terms of predicting rotor performance if these effects were not present.

**Validation of BEMT**

Several of the assumptions that are necessary for application of the simple momentum theory are removed using the BEMT. This
Validation of BEMT against measurements of thrust and power.

For Harrington Rotor 1, includes the ability to calculate tip losses directly for each rotor (i.e., \( \kappa \) need not be assumed, nor assumed as a constant) and also the values of the rotor-on-rotor interference factor \( \kappa_{\text{int}} \) need not be specified a priori. These values may change with operating state, i.e., with net system thrust and with rotor thrust sharing.

Results for the Harrington Rotor 1 and Rotor 2 are shown in Figs. 6 and 7, respectively. In this case, only a value for the viscous drag \( C_d (= 0.011) \) must be assumed. Notice that the BEMT predictions are in substantial agreement with the measurements, and indeed are superior to those obtained previously using the simple momentum theory. This is mainly because the nonuniform radial distribution of induced losses over the disk are being resolved in this case, along with blade tip losses, and so the BEMT gives a better definition of the rotor airloads.

BEMT results for the induced power factor for the isolated rotor versus the coaxial rotor are shown in Fig. 8 for Harrington’s Rotor 1. In this case, the results are shown versus blade loading coefficient \( C_T / \sigma \) rather than \( C_T \). The results show that with tip losses considered, the induced power factor of the single rotor is about 1.10, although this varies slightly with the value of \( C_T / \sigma \). The induced power factor from interference alone, \( \kappa_{\text{int}} \), is about 1.28, and this value is just slightly larger than the result of 1.22 obtained analytically from the simple momentum theory, despite the fact that this rotor system uses untwisted blades. The predicted net induced power factor is about 1.38, and this compares well with the averaged value \( \kappa_{\text{int}} \kappa = (1.28)(1.10) = 1.41 \) found analytically using the simple momentum theory.

Validation of FVM

As already alluded to, a primary problem for the coaxial rotor is that the wake from the upper rotor is ingested into the lower rotor, hence affecting significantly the airloads on the lower rotor. This physical problem is illustrated in Fig. 9, which are wake results from the FVM analysis. Notice that the wake from the upper rotor contracts smoothly into the lower rotor, and its helicoidal structure is preserved as it is convected through and below the lower rotor. This observation is consistent with flow visualization experiments, such as those of Taylor (Ref. 20), where the upper and lower rotor wakes remain distinct. Notice that the wake contraction in this case is about 80% (compared to the ideal value of 70.7%) and is similar in value to that found on isolated rotors (about 78%). This suggests that the spacing between the upper and lower rotor likely has minimal effects on coaxial rotor performance, and that one can proceed on the assumption that for most (all) practical rotor plane spacings the wake from the upper rotor can be considered fully contracted by the time it is ingested by the lower rotor.

The results from the FVM were compared with the thrust and...
power measurements for Harrington’s Rotor 1 and 2, as shown in Figs. 10 and 11. Notice that the FVM predictions are in generally good agreement with the measurements, and also are in good agreement with the results from the BEMT (Fig. 6).

There are few reported spanwise airloads for coaxial rotors, and none in Harrington’s tests. Therefore, spanwise airloads results predicted from the FVM were used to compare with the BEMT. The Harrington Rotor 1 was used, at a net system operating $C_T$ of 0.004 ($C_T/\sigma = 0.0741$) with a torque balance. The results for the inflow distribution predicted by the BEMT and FVM are shown in Figs. 12 and 13 for the upper and lower rotors, respectively. No blade root cutout was used for the BEMT analysis. Notice that the inflow at the lower rotor is substantially affected by the outflow from the upper rotor, with a higher inflow over the inboard regions of the lower rotor. The agreement between the BEMT and FVM is reasonable in both magnitude and form, although less so in detail. However, recall that the BEMT is essentially a 2-D theory compared to the FVM, which is a fully 3-D theory (all three components of the flow field are computed), and so the results are really quite good bearing in mind the extremely low computational cost of the BEMT. Notice that the Prandtl tip loss function gives a particularly good definition of the inflow at the blade tips when compared to the FVM. In this case, for the BEMT predictions the wake from the upper rotor was assumed to contract to $r = 0.82$, consistent with the predictions from the FVM.

The results for the thrust distribution on the coaxial rotor are shown in Figs. 14 and 15 for the upper and lower rotors, respectively. In this case, the agreement between the BEMT and the FVM is very good. For a torque balance the upper rotor carries a greater fraction of the total system thrust ($C_{T,u}/C_{T,l} \approx 1.2$). This is because the induced losses are intrinsically higher on the lower rotor so the upper rotor must operate at a higher thrust (and so with higher torque) to balance the net torque on the system. The effects of the slipstream from the upper rotor can be seen in the thrust gradient, which is substantially reduced inboard compared to the upper rotor. Outside of the region affected by the slipstream, the blade thrust gradient quickly recovers to values that are consistent with those found on the upper rotor. The overall agreement of the two methods is very encouraging, especially bearing in mind that computationally the BEMT is approximately five orders of magnitude faster than the FVM.

The results for the corresponding torque distribution on the coaxial are shown in Figs. 16 and 17 for the upper and lower rotors, respectively. The collective pitch on both rotors was adjusted to give a torque balance, so the net area under the torque curve is equal in both cases. Again, notice that the effect of the slipstream from the upper rotor is to alter the spanwise gradient, especially over the region near the slipstream boundary at 82% span.

Finally, results for the spanwise distribution of local lift coefficient, $C_l$, are shown in Figs. 18 and 19 for the upper and lower rotors, respectively. There is reasonable agreement between the BEMT and the FVM on the upper rotor, with a fairly uniform
$C_l$ distribution, but the BEMT predicts higher lift coefficients inboard and slightly lower lift coefficients outboard. The average lift coefficients are, however, quite adequately predicted. On the lower rotor, the lift coefficient distribution is much less uniform, but here the BEMT and FVM agree fairly well. Notice that for the lower rotor there will be somewhat higher profile losses than are desirable because only the sections near the tip operate at lift coefficients that are near those for their best lift-to-drag ratio.

This point is considered again for the definition of an optimum coaxial rotor.

**Figure of Merit for a Coaxial**

A figure of merit ($FM$) is often used to compare the relative hovering efficiency of different rotors relative to the datum perfor-
mance provided by the simple momentum theory. In application, the concept of an FM is frequently misused because a proviso of its use as an efficiency metric is that it be used to compare rotors that are operated nominally at the same disk loading. This is to avoid unfairly biasing rotors that operate at relatively higher or lower disk loadings and so with different relative contributions of induced and profile losses. The problem of defining an FM for a coaxial rotor, therefore, is not an obvious one because each rotor of the coaxial system typically operates at an unequal thrust (hence unequal disk loading) to enable a torque balance. Fundamentally, any definition of FM can be adopted, as long as the definition is used to compare like-with-like, i.e., to compare a single rotor with another single rotor or a coaxial with another coaxial at equivalent disk loadings.

One definition that has been used for the FM of a coaxial rotor is based on an isolated single rotor operated at the same net system thrust, W, or in coefficient form: \( C_W = C_{T_0} + C_T \), i.e.,

\[
FM = \frac{C_W^{3/2}}{\kappa_{int} \kappa W^{3/2}} + \frac{N_r \sigma C_{d_0}}{8} \frac{\sqrt{\rho A}}{\sigma} (\Omega R)^3
\]  

(23)

where \( \kappa_{int} = 1 \) for an isolated single rotor and \( \sigma \) is the solidity for one rotor. In dimensional terms this is equivalent to the definition

\[
FM = \frac{W^{3/2}}{\kappa_{int} W^{3/2}} + \frac{N_r \sigma C_{d_0}}{8} \frac{\rho A (\Omega R)^3}{\sigma}
\]  

(24)

This is the definition unilaterally used by Coleman (Ref. 6) in his review article. As long as all results are computed using this definition, the FM of different coaxial rotors can be compared. It would not, however, be entirely proper to compare the values of the FM for single and coaxial rotors using this definition.

Results for the FM obtained using this (Coleman’s) definition for the two Harrington rotors are compared in Figs. 20 and 21 along with predictions using the BEMT. Bearing in mind the agreement found for the performance polars (Figs. 6 and 7), the correlation with the corresponding FM values is not surprising. On the basis of this comparison it would be easy to conclude that the coaxial rotor initially has a higher FM than a single rotor when operated at the same equivalent blade loading coefficient, but that the maximum FM for both systems are nearly the same. This is perhaps surprising bearing in mind that it has already been established by means of the momentum theory that the coaxial rotor suffers from higher induced losses (and so lower net efficiency) when compared to two isolated rotors.

However, Coleman’s definition of the FM for a coaxial does not reflect the fact that there are two rotors that share the net system thrust, and so each rotor actually operates at a much lower (about half) disk loading. Alternatively, this is equivalent to comparing rotors at different disk loadings, which violates the assumptions inherent to the definition of the FM in the first place. This is because increasing the disk loading, \( DL (= T/A) \) will increase the induced power relative to the profile power, producing a higher value of the FM and a potentially misleading comparison between two different rotors. This can easily be seen if the FM for a single rotor is written dimensionally as

\[
FM = \frac{P_{ideal}}{\kappa P_{ideal} + P_0} = \frac{1}{\kappa + \frac{P_0}{P_{ideal}}} = \frac{1}{\kappa + \sqrt{\frac{2P_0}{T}} \sqrt{DL}}
\]  

(25)

\[
WM = \frac{W^{3/2}}{\kappa_{int} W^{3/2}} + \frac{N_r \sigma C_{d_0}}{8} \frac{\rho A (\Omega R)^3}{\sigma}
\]  

(26)

or in non-dimensional terms then

\[
FM = \frac{C_W^{3/2}}{\kappa_{int} W^{3/2}} + \frac{N_r \sigma C_{d_0}}{8} \frac{\rho A (\Omega R)^3}{\sigma}
\]  

(27)

Because the thrust sharing between each rotor of the coaxial system may not be known (such as in Harrington’s experiments), either of the two previous definitions (Eq. 23 or Eq. 27) are the only definitions that can be used in this case. However, with this
alternative definition of the \( FM \), the true efficiency of the coaxial can be better compared with the single rotor.

Results using this alternative definition of the \( FM \) are shown in Fig. 22, where the actual reduced efficiency of the coaxial rotor system now becomes apparent. In this case, the \( FM \) for the upper and lower rotors have been calculated separately. While the upper rotor achieves a good individual \( FM \), the lower rotor has a much smaller \( FM \) because it works in the outflow of the upper rotor and so it incurs higher induced losses. With these results, it is apparent that the \( FM \) of the system should not be higher in value than the \( FM \) of the upper rotor, although Coleman’s definition of the \( FM \) suggests this to be the case.

The fault lies in the fact that the effective disk loading of the system is assumed to be twice that of either of the two rotors, therefore, elevating the value of the \( FM \) and so violating the assumptions inherent in the definition of the \( FM \) as an efficiency metric. By using the proper definition of the \( FM \) based on thrust sharing between the rotors, clearly the coaxial achieves a somewhat lower maximum figure of merit compared to a single rotor, all other factors being equal (i.e., when compared at the same disk loading, net solidity and tip speed), which is more intuitively correct anyway. Because the lower rotor of a coaxial operates in the slipstream of the upper rotor, its net induced velocity is always higher for a given value of system thrust and \( \kappa_{int} > 1 \) (where the values of \( \kappa_{int} \) depend on the assumptions made), and so the average system efficiency is always lower compared to two isolated rotors operated at the same disk loading.

A more precise definition of the \( FM \) for a coaxial rotor system must consider the relative thrust sharing and generally unequal disk loadings of upper and lower rotors. In this case the ideal power for two rotors free of interference effects can be written as

\[
C_{P_{ideal}} = \frac{C_{T_2}^{3/2}}{\sqrt{2}} + \frac{C_{T_1}^{3/2}}{\sqrt{2}}
\]

which reflects the unequal disk loadings of the two rotors. This result can be rewritten in terms of the thrust sharing ratio \( C_{T_2}/C_{T_1} \) as

\[
C_{P_{ideal}} = \frac{C_{T_2}^{3/2}}{\sqrt{2}} \left[ \frac{C_{T_1}}{C_{T_2}} \right]^{3/2} + 1
\]

from which the \( FM \) follows as

\[
FM = \frac{C_{T_2}^{3/2}}{\sqrt{2}} \left[ \frac{C_{T_1}}{C_{T_2}} \right]^{3/2} + 1
\]

The previous definition of the \( FM \) then follows as a special case of \( C_{T_2} = C_{T_1} \). In the case of a torque balance the upper rotor generally always carries a greater fraction of the total thrust than the lower rotor so that \( C_{T_2}/C_{T_1} > 1 \). In practice, the net effect is that the \( FM \) of a coaxial using this definition is just slightly higher than that obtained assuming both rotors carry an equal fraction of the total thrust.

**Optimum Hovering Coaxial Rotor**

The results from the FVM and BEMT have shown such generally good agreement that the BEMT by itself forms a rapid and cost-effective basis for finding the initial optimum rotor design. In the case of a single helicopter rotor, Gessow (Ref. 22) showed that the optimum design must use hyperbolically twisted blades to obtain uniform inflow and minimum induced losses. This result must be combined with a hyperbolically tapered blade chord to produce a condition where all the airfoil sections on the blades also operate at their best lift-to-drag ratios (or close to), and so to produce minimum profile losses. The optimum geometric blade design, however, is not realizable in practice, but it can be closely approximated using linear twist and taper distributions. The goal is now to seek the equivalent optimum blade design for the coaxial rotor system.

The solution for the inflow over the upper and lower rotors is given by Eqs. 15, 18 and 19. Other physical factors that can affect coaxial rotor performance include a thrust (or power) recovery effect through the removal of swirl losses in the downstream, although this effect is important only at very high values of disk loading such as used on propellers, and so can be justly neglected as an initial approximation for the optimum coaxial rotor design. On the upper rotor, it is rather obvious that the inflow distribution for minimum induced losses will be uniform. This is equivalent to producing a uniform disk loading and a linear thrust gradient from zero at the rotational axis to a maximum value at the blade tip (assuming no tip losses), i.e., \( dC_T/dr = (\text{constant})/r \). Therefore, Eq. 15 can be solved for the distribution of \( \theta_u(r) \) that will achieve this goal. It is readily apparent from the equation in this case that the blade twist distribution must be hyperbolic, i.e., \( \theta_u = \theta_{hyp}/r \).

On the lower rotor, the inflow distribution cannot be made uniform, at least not without leading to gross aerodynamic inefficiency. In fact, it has already been shown that for Harrington’s Rotor 2 that the high inflow on the inner part of the lower rotor can produce reduced thrust inboard and a departure from the ideal thrust distribution (which is linear). However, the distribution of inflow for minimum losses is the condition where the inflow inside the region affected by the upper rotor is made uniform and the inflow is also made uniform outside this region, i.e., both regions now have uniform inflow but not of the same value. This is consistent with the simple momentum theory solution that has been discussed previously. The optimum (minimum induced power) condition in this case, however, is still uniform disk loading and a linear thrust distribution consistent with the isolated rotor case. The overall problem for the coaxial must also
be solved consistently with either a thrust or torque balance on the upper and lower rotors.

The procedure to find the optimum coaxial rotor using the BEMT as a basis can be undertaken numerically. The process starts with the specification of the net system thrust \( C_T = C_W \) and the requirement for either an equal thrust sharing \((C_{Tu} = C_{Tl})\) or net torque balance \((C_{Qu} - C_{Ql} = 0)\) on the two rotors. The loads on the upper rotor are determined from any initially presumed distribution of \( \theta_u(r) \) and so finding a first solution for \( \lambda(r) \). A lift-curve slope, \( C_{l\alpha} \), of \( 2\pi \) per radian can be assumed, although \( C_{l\alpha} \) can be specified empirically and the linearized effects of compressibility can be accounted for using the Glauert correction at each blade station. The thrust distribution \( dC_{Tu}/dr \) is then determined and the net rotor thrust \( C_{Tu} \) is found by radial integration.

The goal of finding \( \theta_u \) for a linear thrust distribution \( dC_{Tu}/dr = \text{(constant)} \) was accomplished by means of a fixed-point iteration using Eq. 15; the assumption of radial independence allows \( \theta_u \) to be determined section-by-section, which as mentioned previously, has good validity except at the blade tips. The rotors are trimmed initially to carry equal thrust by further adjusting the reference blade pitch on the upper rotor by means of a Newton–Raphson iteration.

The solution for the upper rotor then dictates its outflow in the vena contracta, and so the upstream flow into the lower rotor. The process of finding \( \theta_u \) is then accomplished by numerically solving Eqs. 18 and 19 for \( \lambda \), while imposing the same goal of a linear thrust distribution and uniform disk loading, i.e., in this case \( dC_{Tu}/dr = \text{(constant)} \). This requirement gives uniform disk loading on the lower rotor and uniform inflow distributions over the inner and outer regions of the lower rotor disk. Again, the reference blade pitch is adjusted iteratively as part of the initial solution to give equal thrust sharing \( C_{Tu} = C_{Tl} = C_T/2 \). The process continues by finding \( \theta_l \) to give a torque balance on the two rotors (finding \( C_{Qu} - C_{Ql} = 0 \) to within a 0.05% error is sufficient), which is accomplished by a Newton–Raphson iteration. The process converges smoothly and quickly, although at least 100 radial points must be used to accurately define the spanwise airloads over the disk less the gradients be inadequately resolved.

A representative set of results from the optimal coaxial rotor are shown in Figs. 23 through 25. In this case, Harrington’s Rotor 2 (zero blade twist and no taper) was considered as the baseline rotor, mainly because the improvements in relative performance can be established relative to the known (measured) results. In the design for the optimum rotor, it was designed to operate at a net system \( C_T \) of 0.016 at a torque balance. An ideal wake contraction of 70.7% (i.e., \( A/A_r = 2 \)) from the upper rotor was assumed in this case, although the contraction will not be as large as this in practice, as previously discussed.

Figure 23 shows the inflow distribution over the upper and lower rotors for an optimum coaxial rotor configuration. Harrington’s Rotor 2 used as baseline.

Figure 24: Distribution of thrust on upper and lower rotors for an optimum coaxial rotor configuration. Harrington’s Rotor 2 used as baseline.

Figure 25: Blade twist distribution on the upper and lower rotors for an optimum coaxial rotor configuration. Harrington’s Rotor 2 used as baseline.

Figure 26: Distribution of torque on upper and lower rotors for an optimum coaxial rotor configuration. Harrington’s Rotor 2 used as baseline.
lower rotors. Notice that for each rotor the inflow is uniformly distributed, as required for uniform disk loading, but clearly there is a much higher net inflow on the inboard part of the lower rotor, which of course is induced by the slipstream from the upper rotor.

The linear thrust distribution shown in Fig. 24 confirms that uniform disk loading has indeed been achieved through the numerical optimization of the blade twist. The need for a torque balance means that there is a different thrust on the upper and lower rotors; the larger induced losses on the lower rotor means that the lower rotor must have a lower thrust than the upper rotor. This load sharing depends on the net system thrust as well as the individual rotor design(s), so the blade design for optimal hovering performance is nonunique but qualitatively the same.

The blade twist distribution (Fig. 25) to achieve the ideal loading (uniform disk loading on each rotor) is of particular interest. While hyperbolic twist on the upper rotor is an expected result, notice that a double-valued form of hyperbolic twist is required on the lower rotor to compensate for the outflow from the upper rotor. Clearly the amount of blade twist required is considerable, but this is needed to correct for the degraded lift and non-linear lift distribution that would otherwise be produced by the high inflow from the slipstream induced by the upper rotor. The blade pitch required to produce useful lift is only slightly larger than for the upper rotor, bearing in mind there is a torque balance on the coaxial system and that a higher blade pitch angle is required on the upper rotor. Outside the slipstream boundary, the blade pitch requirements are reduced.

The torque gradient in Fig. 26 shows that the steeper torque gradient obtained on the inner part of the lower rotor from the slipstream of the upper rotor is offset by the outer part of the lower rotor – a torque balance is being obtained in this case, so the net area under both torque curves are equal (i.e., $C_Q = C_{Q}^d$).

The distribution of local lift coefficient on the upper and lower rotors is shown in Fig. 27. It is apparent from Fig. 24 that upper rotor must carry a higher proportion of the total system thrust to achieve a torque balance, therefore, the upper rotor operates at a higher average lift coefficient than the lower rotor. This latter observation leads to two consequences. First, the upper rotor operates at higher angles of attack and so at lift coefficients and lift-to-drag ratios that are different than the lower rotor. To minimize profile losses, both rotors will need the blades to operate at their best $C_l/C_d$ and this condition cannot be met precisely if both rotors operate at different mean lift coefficients. While this difference is relatively small inasmuch that in practice the $C_l/C_d$ curves of most airfoil sections are relatively flat over a broad range of angles of attack, a true optimum coaxial rotor must still have all blade stations on both rotors operating as closely as possible to their best values of $C_l/C_d$.

Second, it is apparent that the upper rotor will dictate the stall margins of the rotor as a whole, and so it will ultimately limit the net performance of the rotor system. One way to rectify this problem for an optimum coaxial rotor is to slightly increase the solidity of the upper rotor and so reduce the average lift coefficients there and slightly decrease the solidity of the lower rotor. While the practical construction of two rotors with different blade designs (both in terms of twist and solidity) will be more expensive than a rotor system using identical blades throughout, this is the price to pay for truly optimum coaxial rotor performance.

The hovering performance of the optimal coaxial rotor relative to the baseline Harrington Rotor 2 is shown in Fig. 28 using the BEMT, with the corresponding $FM$ plot being shown in Fig. 29. Clearly the use of optimal blade twist (in this case for net system $C_T = 0.008$) gives some reduction in rotor power and an increase in $FM$ at the design point, and also for higher values of $C_T$. The gains, however, are modest because the primary losses in a coaxial rotor system stem from the rotor-on-rotor interference, and the induced losses cannot be reduced (through an optimum blade design) any further than the value given by the simple momentum theory. However, the use of optimum blade twist (or something close to the hyperbolic form on both rotors)
causes both rotors to operate more efficiently over their entire blade span.

The reductions in net induced power factor (the product $\kappa \kappa_{int}$) are confirmed with the results in Fig. 30, with a reduction from about 1.45 in the baseline case to about 1.37 in the optimum case. Recall that the theoretical minimum will be $\kappa_{int} = 1.22$ (no tip losses) and with an induced factor of 1.1 with tip losses this brings the total induced power factor to $(1.22)(1.1) = 1.34$, which is consistent with the numerically estimated value from the BEMT.

**Optimum MTR Proprotor**

An initial optimum design for the MTR’s proprotor has been considered using the aerodynamic principles previously outlined. In the first instance, the static thrust (i.e., hovering) efficiency of the coaxial proprotor system has been addressed. This is followed by a discussion of an optimum design for propulsive efficiency. By means of a separate design analysis of a variant of the MTR, a proprotor with 5 blades per rotor with a net solidity of 0.129 at an operating $C_T$ of 0.016 was considered as an initial design point.

The blade twist to achieve optimum hovering efficiency of the proprotor is shown in Fig. 31. Recall that the rotor system was designed for a torque balance. Clearly the amount of blade twist required is considerable, with a larger amount of twist approaching 40 degrees (between the tip and an assumed root cut-out at $r = 0.2$) on the lower rotor. Notice that a larger amount of twist on the blades of the lower rotor, however, is needed to compensate for the high outflow induced by the upper rotor, so the blade pitch required to produce useful lift is somewhat larger than for the upper rotor. Outside the slipstream boundary, the blade pitch requirements are reduced.

The figure of merit of the optimum hovering proprotor is shown in Fig. 32. Notice that the results are presented for the upper rotor, lower rotor, and for the combined coaxial rotor system. A maximum net $FM$ of 0.62 is consistent with a typical coaxial rotor system. Notice that the calculations have been carried out until the rotor(s) stall. The onset of stall causes a loss in rotor thrust and an increase of power requirements, substantially reducing $FM$ and causing the $FM$ curve to double back on itself. Clearly the definition of $FM$ based on system thrust produces a ficticiously high value of overall system efficiency.

An alternative presentation is to examine the $FM$ versus blade loading coefficient, $C_T/\sigma$, as shown in Fig. 33. Because the values of $C_T/\sigma$ are proportional to the average lift coefficient on the upper and lower rotors, this plot shows how the maximum achievable $FM$ of the coaxial as a system is being set by the onset of stall on the upper rotor, although only marginally so in this case. On the upper rotor, the onset of stall occurs at $C_T/\sigma \approx 0.14$ whereas on the lower rotor, the effects become apparent at a slightly higher $C_T/\sigma$ of 0.155. This is reflected in
the $FM$ curve of each individual rotor. The point here is that stall margins for the coaxial rotor system as a whole are generally going to be set by the stall margins of the upper rotor, and suggests that to maximize overall stall margins on an optimum coaxial rotor system, the upper rotor will need to have a slightly larger solidity than the lower rotor.

**Axial Flight Performance**

The axial flight (high inflow) performance of the MTR’s coaxial rotor was first examined using the BEMT as a function of tip speed ratio. The hover condition is a limiting case of zero axial flow velocity. The tip speed ratio is defined as the ratio of the true axial flow airspeed, $V_\infty$, to the tip speed of the rotor, $\Omega R$.

Of primary interest is the propulsive efficiency of the coaxial rotor system. This can be defined in the usual way as

$$\eta = \frac{\text{Ideal propulsive power}}{\text{Actual shaft power required}} = \frac{TV_\infty}{TV_\infty + P_i + P_0} \tag{31}$$

where $T$ is the net system thrust, $P_i$ is the induced power and $P_0$ is the profile power. The induced power becomes a smaller fraction of the total power requirements with increasing $V_\infty$, so that for higher tip speed ratios the minimization of profile power losses becomes an important consideration in any proprotor design for high aerodynamics efficiency.

A series of calculations were conducted with the BEMT to evaluate axial flight performance of the MTR’s coaxial rotor. The performance was calculated for a series of increasing values of reference blade pitch (collective pitch) over a range of tip speed ratios.*

Results for the system thrust coefficient are shown in Fig. 34 for a coaxial rotor system with the blade twist derived for hovering flight under the assumption of an upper wake contraction ratio of 0.82. This rotor system is designed for hovering flight (i.e., to produce a high static thrust efficiency) and not for high propulsive efficiency in axial flight. The initial range of acceptable thrust performance is bounded by blade stall (high blade pitch at low tip speed ratios) and the generation of negative thrust (higher tip speed ratios at lower blade pitch). The relationship between $C_T$ and tip speed ratio is almost linear in the normal working state of the proprotor.

For proprotors that initially operate with large blade pitch at low tip speed ratios, the blades will always be stalled. The flow will then attach to the blades as tip speed ratio increases, and $C_T$ will then decrease almost linearly with increasing tip speed ratio until negative thrust conditions (the brake state) are encountered at a fixed blade pitch. For higher tip speed ratios the performance of the proprotor is degraded because of increasing compressibility losses. At the tip of the blade, the incident Mach number is defined by the resultant helical velocity of the blade. Therefore, compressibility effects become important when the helical Mach number approaches and begins to exceed the drag divergence Mach number of the airfoil sections being used.

What is apparent in this case is that blade stall and/or compressibility effects limit the performance of the proprotor system at relatively low tip speed ratios, and at substantially lower conditions than would be needed for efficient performance of the MTR in axial flight. This is better shown in Fig. 35, which is a plot of propulsive efficiency versus tip speed ratio. Notice that characteristic propeller efficiency plots are obtained at low collective pitch and low tip speed ratios. The curves show a rapid increase in efficiency to a maximum, followed by a sharp drop as $C_T$ decreases above a certain value of tip speed ratio. At higher collective pitch, blade stall substantially reduces proprotor efficiency and the efficiency curves follow the blade stall boundary until a sufficiently high tip speed ratio is reached that the flow attaches to the blades. The efficiency curves show an increase until performance is limited by either drag rise from compressibility or because a torque balance between the upper and lower rotors cannot be achieved. In this particular case, these limits are reached at relatively low tip speed ratios (0.3 to 0.4).

The problem of limiting aerodynamic performance can be seen by examining the separate propulsive efficiency of the upper and lower rotors, as shown in Figs. 36 and 37. The propulsive efficiency of the upper rotor is defined by

$$\eta_u = \frac{T_u V_\infty}{T_u V_\infty + P_i + P_0} \tag{32}$$

and for the lower rotor by

$$\eta_l = \frac{T_l V_\infty}{T_l V_\infty + P_i + P_0} \tag{33}$$

---

*Propeller performance is usually plotted versus “advance ratio” $J$, which is defined as $V_\infty/(nD)$ where $n$ is the propeller speed in terms of revolutions per second and $D$ is the propeller diameter. The values of “tip speed ratio” and $J$ are related simply by: tip speed ratio $= J/\pi$. 

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**Figure 34:** Predicted variation of the rotor system thrust versus tip speed ratio for the baseline “ideal” MTR rotor optimized for hovering performance.

**Figure 35:** Predicted variation of propulsive efficiency versus tip speed ratio for baseline “ideal” MTR rotor optimized for hovering performance.
Clearly the upper rotor behaves quite typically in terms of performance until higher collective pitch values are reached, and then the net system efficiency is dictated by the propensity of the lower rotor to remain stalled until a tip speed ratio of about 0.2 is reached. When the lower rotor does unstall, the propulsive efficiency of the system increases rapidly. However, as the efficiency of the lower rotor increases, the upper rotor is reaching its maximum efficiency and further performance is limited by decreasing thrust and compressibility losses on the upper rotor. Beyond this point a torque balance between the rotors cannot be achieved and so the performance of the coaxial rotors as a system, therefore, becomes limited.

It is not immediately obvious that the need for a torque balance will limit the performance of a coaxial rotor system. This is better seen by plotting the thrust and propulsive efficiency of the isolated upper rotor, which are shown in Figs. 38 and 39, respectively. Notice that the isolated upper rotor can operate efficiently over a much wider range of operating conditions than when it is aerodynamically coupled to the lower rotor through the need for a torque balance (compare to Fig. 36). In the case of the isolated rotor, efficient operation is possible up to tip speed ratios exceeding 0.5. This is compared to the coaxial proprotor as a system, where efficient operation is possible only up to tip speed ratios of between 0.3 and 0.4, beyond which the loss of efficiency of one or other of the upper or lower rotors, or the inability to achieve a torque balance, limits the net propulsive efficiency of the system. This problem by itself shows how the “optimum” design of a coaxial proprotor is considerably more difficult than for a single proprotor.

Optimal Proprotor in Axial Flight

It is now apparent that a coaxial proprotor system designed for maximum hovering (static thrust efficiency) can never function efficiently as a propulsor in axial flight over the required range of thrusts and tip speed ratios, so that a new blade design solution must be sought. Again, this design must be obtained based on the need to always maintain a torque balance between the upper and lower rotors, which contains the number of possible blade geometric solutions.

One version of the MTR is designed to cruise at about 200 kts and 20,000 ft. Under these conditions, the thrust for the rotor is substantially lower than that required for hovering flight (the lift-to-drag ratio of the MTR is greater than 10 in cruise). For an initial axial flight design, an “ideal” rotor was designed for a net system $C_T$ of 0.008 at a tip speed ratio of 0.5. The wake contraction ratio from the upper rotor under these conditions was found (from the FVM) to be substantially smaller at a ratio of 0.95 than found for hovering flight (0.82) – see Figs. 40 and 41 – so a wake contraction ratio of 0.95 was used in the design optimization.
In axial flight, the optimum aerodynamic solution for lowest induced losses is still uniform disk loading on both the upper and lower rotors. However, in axial flight the induced velocity becomes a much smaller part of the total inflow through the rotor system. Therefore, the ideal distributions of blade pitch (twist) on the upper and lower rotors will be expected to be more similar. However, the magnitude and distribution of the blade pitch is also expected to be substantially different to the hover case. This result is shown in Fig. 42, where the “ideal” blade pitch distributions are plotted for both the hover and axial flight conditions at the respective design points. Recall that the assumed wake contraction in hover and axial flight are different (0.82 and 0.95, respectively), so the jump or discontinuity in the blade twist distribution on the lower rotor occurs at different spanwise locations. Notice also that because of the smaller induced velocity through the rotor in axial flight, the jump in blade pitch at the outer span is substantially lower for the axial flight case. This latter point is important when it comes to the compromise in blade design that needs to be made to match the requirements of efficient hovering flight (high $FM$) and efficient axial flight (high $\eta$).

The performance of this axial flight optimized proprotor design is shown in Fig. 43. Notice that the proprotor operates efficiently around the design point (tip speed ratio of 0.5) as would be expected. Propulsive performance degrades for tip speed ratios above 0.7, mainly because of compressibility losses and the onset of shock induced flow separation (drag rise) resulting from the high helical tip Mach numbers. These effects, of course, are ameliorated somewhat by reducing rotor speed ($\Omega R$), but this also increases tip speed ratio somewhat. Nevertheless, the operational performance and efficiency of the coaxial proprotor is substantially better than that found for the baseline (hover) design, but recall that this is still basically a point design.

In this case, the upper and lower rotors operate in a more equal aerodynamic environment so that their relative propulsive efficiencies are very similar, as shown in Figs. 44 and 45. While the upper rotor still demonstrates a higher relative propulsive efficiency, and also a better operational efficiency over a wider range of conditions (i.e., collective pitch values and tip speed ratios), the off design performance is still limited.

This “optimized” axial flight design, however, still provides a decent level of hovering performance, as shown in Fig. 46, although the range of conditions where a torque balance without stall can be obtained is substantially reduced. Stall margins for this rotor are also significantly reduced, with the overall immediate conclusion that this rotor system will not be acceptable to meet both efficient axial flight and hovering flight requirements.
Hybrid Optimum Blade Design

The foregoing results show that a “hybrid” coaxial rotor blade design will be necessary to meet both hovering and axial flight requirements. In the first instance, a hybrid blade twist that was a weighted average between the optimum (“ideal”) found for hover and axial flight was examined. While this approach is not a true aerodynamic optimum in the sense that the blade pitch is designed from the onset to meet the thrust and efficiency requirements of both hover and axial flight while also maintaining a torque balance, the results subsequently obtained were sufficiently convincing to justify the approach, at least as a first step toward the hybrid optimum blade design.

Figure 47 shows one example of a hybrid blade pitch design, which in this case is equally geometrically weighted between the optimum blade twist values for the hover and axial flight cases. First, notice the hybrid design has a lower blade twist rate per unit length than for the optimum hovering case. Second, the step change in the twist rate on the lower rotor is substantially reduced, although now two smaller step changes in blade twist are produced. However, these step changes in twist can be approximated by a linear distribution between the two points for manufacturing purposes.

The hovering performance of the rotor system with the hybrid blades is shown in Fig. 48. While showing a small decrease in maximum system figure of merit relative to the rotor system optimized for hovering flight, the $FM$ is substantially better than for the rotor system that was optimized for axial flight alone. The net decrease in stall margins, while significant, is still acceptable.

The most noticeable improvements with the hybrid blade twist design occurs in axial flight. Results for the propulsive efficiency versus tip speed ratio are shown in Fig. 49, respectively. Notice the much wider range of acceptable operating conditions compared to either the baseline hovering design (Fig. 35) or the single point axial flight design (Figs. 43). This is because the upper and lower rotors of the coaxial system now demonstrate a more balanced performance, as shown in Figs. 50 and 51, with neither of the upper or lower proprotors by themselves significantly limiting system performance from stall or compressibility losses. The onset of stall at higher blade pitch and low tip speed ratios is apparent, as shown by the large decrease in rotor efficiency. This is because the onset of compressibility losses associated with the higher helical Mach numbers at higher tip speed ratios leads to significant increases in power required.

A net average propulsive efficiency of 80% up to a tip speed
ratio of 0.6 is considered acceptably good performance, although further improvements may be possible. To this end, the primary improvements will be through blade planform changes to more optimally distribute the local lift coefficients on the upper and lower rotors so that they operate at (or closer to) their best lift-to-drag ratios. As previously alluded to, the distribution of lift coefficients over the blade are of significant importance in the design of an optimal proprotor system. Representative results for hovering flight are shown in Fig. 52. The need for low induced power losses results in high blade pitch inboard on the blades, and so high lift coefficients are produced. This means that some taper of the blade planform is required to reduce inboard lift coefficients so that they remain below stall at the design condition, and also so that the blade sections operate closer to their local maximum lift-to-drag ratio to minimize profile losses.

Notice from Fig. 52 that some taper is desirable in reducing inboard lift coefficients; a 2:1 taper provides an acceptable reduction so that the inboard blade sections will remain below stall. Although much depends on the airfoils section that is ultimately chosen for the inboard region, it is realistic to expect maximum lift coefficients of 1.65 with a state-of-the-art airfoil design. With a typical root cut-out region of 15% of blade radius, the proprotor should not become stall limited at its design condition.

Notice also from Fig. 52 that the upper and lower rotors operate at different average lift coefficients, another point alluded to previously. This is because the need for a torque balance means that the upper rotor generates a higher fraction of the total system thrust. This means that the upper rotor will dictate the stall margins for the coaxial rotor system as a whole, and the rotor system will be more limited in its performance than if both rotors simultaneously reached their stall limits. This can only be
achieved if the average weighted solidity for the upper rotor is slightly increased and the solidity for the lower rotor is slightly decreased. While for a sub-scaled flight demonstrator this may not be necessary because margins are expected to be more generous, it may become a serious issue for the full-scale MTR if rotor weight is to be minimized.

**Composite Efficiency Metric**

The efficiency of a coaxial proprotor has now been evaluated as two separate metrics: 1. A figure of merit in hover and, 2. A propulsive efficiency in forward flight. The primary difficulty of the figure of merit is that it has no validity in axial flight. The difficulty with the propulsive efficiency as a metric is that it approaches zero for low tip speed ratios, and so gives a false impression of efficiency at low axial flight speeds.

A composite efficiency can be defined in terms of the ratio of ideal power required for flight to the actual power required. For a coaxial rotor system in a torque balanced situation, the upper and lower rotors carry an unequal portion of the total thrust, so any valid efficiency metric must take this behavior into account. In light of this, a composite efficiency metric for a coaxial proprotor can be defined as

\[
\eta_c = \frac{\text{Ideal power}}{\text{Actual shaft power required}} = \frac{TV_\infty + P_{\text{ideal},u} + P_{\text{ideal},l}}{TV_\infty + P_u + P_0 + P_l} \quad (34)
\]

where \(P_{\text{ideal},u}\) represents the ideal power required for the isolated upper rotor and \(P_{\text{ideal},l}\) is the ideal power for the isolated lower rotor. The terms \(P_u\) and \(P_0\) are respectively the actual induced and profile power values for the upper rotor and \(P_l\) and \(P_0\) are the corresponding values for the lower rotor. In terms of non-dimensional coefficients, \(\eta_c\) can be written as

\[
\eta_c = \frac{C_T\lambda_\infty + C_{P_{\text{ideal},u}} + C_{P_{\text{ideal},l}}}{C_T\lambda_\infty + C_{P_u} + C_{P_0} + C_{P_l} + C_{P_l}} \quad (35)
\]

The ideal power is a function of the tip speed ratio, and can be calculated on the basis of simple momentum theory for a rotor in axial flight (i.e., uniform inflow assumptions and no losses other than induced losses). For the upper rotor the ideal power coefficient can be written as

\[
C_{P_{\text{ideal},u}} = C_T \left(\frac{1}{2} \sqrt{\lambda_\infty^2 + 2C_T} - \frac{\lambda_\infty}{2}\right) \quad (36)
\]

and for the lower rotor it is

\[
C_{P_{\text{ideal},l}} = C_T \left(\frac{1}{2} \sqrt{\lambda_\infty^2 + 2C_T} - \frac{\lambda_\infty}{2}\right) \quad (37)
\]

where \(\lambda_\infty\) is the tip speed ratio, \(V_\infty/\Omega R\).

The results for the proprotor with the hybrid blade twist distributions is shown in Fig. 53. Notice that in this case as tip speed ratio approaches zero the composite efficiency approaches the figure of merit for hovering flight (see Fig. 48). At higher tip speed ratios, the composite efficiency approaches the propulsive efficiency (see Fig. 49). Therefore, this composite efficiency provides a single metric from which to evaluate the performance of candidate coaxial proprotor designs.

**Conclusions**

This paper has has outlined some fundamental aerodynamic design and performance issues that are associated with the design of a coaxial proprotor. In summary, the following observations and conclusions have been drawn from this study:

1. The fundamental problem in the design of an efficient coaxial rotor system is to account accurately for the aerodynamic interference between the rotors and, in particular, to model the aerodynamic influence of the upper rotor on the lower rotor at a proper torque balance.
2. The aerodynamic performance of the coaxial rotor was initially approached using the simple momentum theory analysis. The minimum induced losses for a coaxial system is obtained when the lower rotor operates in the fully developed slipstream of the upper rotor and at equal and opposite torque to the upper rotor.
3. The simple momentum theory was validated against thrust and power measurements for a coaxial rotor system, with fairly good agreement. However, induced interference and tip loss effects must be assumed a priori. Measurements show that the lower bound of the interference effects suggested by momentum theory provides a good representation of the actual rotor-on-rotor losses.
4. A definition of a figure of merit based on total system thrust was shown not to be the best metric for comparison between different rotor systems. A new figure of merit expression for a coaxial rotor was established based on a thrust sharing ratio between the upper and lower rotors. This reflects the case that each rotor of the coaxial system operates at a different disk loading and so at a different individual aerodynamic efficiency.
5. The blade element momentum theory (BEMT) for a coaxial rotor was formally established. This theory was implemented numerically and used to solve for the distribution of local airloads over the upper and lower rotors. The BEMT was found to agree very well with measured coaxial rotor performance, and gives better results than the simple momentum theory.
6. The results from the BEMT were also validated using a free-vortex wake analysis of the coaxial rotor. Despite the much higher computational cost of this method, the results...
suggest that both methods will be useful in the design of an optimum performance coaxial rotor system.

7. The ideas of an optimum coaxial rotor system have been discussed. The optimum case for minimum induced losses occurs when both the upper and lower rotors operate with a uniform disk loading at a balanced torque. This result corresponds to linear distributions of local thrust over the blades of both the upper and lower rotors. The ideal case also corresponds to uniform inflow on the upper rotor but a double-valued uniform inflow on the lower rotor.

8. The BEMT was used to design an optimum rotor for minimum losses and maximum figure of merit. The optimum twist for the lower rotor corresponds to a double hyperbolic twist, with the break point being at the location where the wake from the upper rotor impinges the lower rotor. Therefore, the design of a truly optimum coaxial rotor will require close attention to the use of the proper amount of blade twist on both the upper and lower rotors.

9. A by-product of the coaxial rotor optimization shows that the upper and lower rotors operate at different thrust and mean lift coefficients. Separate optimization of both blade planforms will be required to produce distributions of lift coefficients that minimize profile losses by having all the blade sections work close to their best lift-to-drag ratios.

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References


Appendix

Case 1 – Assume that the rotor planes are sufficiently close together (Fig. 1) and that each rotor provides an equal fraction of the total system thrust \((W = 2T)\) where \(T_u = T_l = T = W/2\). The effective induced velocity of the dual rotor system will be

\[
(v_i)_e = \sqrt{\frac{2T}{2\rho A}} = \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{W}{2\rho A}} \tag{A1}
\]
where $A$ is the disk area of any one rotor. Therefore, the induced power, $P_i$, is

$$ (P_i)_{\text{coax}} = 2T (v_i) = 2T \left( \frac{2T}{2pA} \right)^{3/2} = 2 \sqrt{2} \rho \frac{T^{3/2}}{2pA} \quad (A2) $$

The basis on which to compare this result is to consider each rotor as operating separately as free, isolated rotors. The induced power for either rotor will be $T v_i$ and for the two separate rotors the induced power is

$$ P_i = 2T \sqrt{\frac{2}{2pA}} \quad (A3) $$

If the interference-induced power factor $\kappa_{\text{int}}$ associated with the coaxial system is considered to be the ratio of the results in Eqs. A2 and A3 then

$$ \kappa_{\text{int}} = \frac{(P_i)_{\text{coax}}}{(P_i)_{\text{isolated}}} = \left( \frac{2}{\sqrt{2}} \right)^{3/2} \sqrt{\frac{2}{2pA}} = \sqrt{2} \quad (A4) $$

It is of immediate significance here to note that there is a 41% increase in induced power relative to the power required to operate the two rotors in complete isolation. The net power of the coaxial rotor system can, therefore, be written as

$$ (P_i)_{\text{coax}} = \kappa_{\text{int}} \left( \frac{2T}{2pA} \right)^{3/2} = \kappa_{\text{int}} W^{3/2} \quad (A5) $$

A corollary to this simple result is that for coaxial rotors that rotate in the same plane (or very nearly so in practice), the induced power factor is independent of the thrust sharing between the rotors (i.e., when $T_u \neq T_l$). This can be seen by writing

$$ \kappa_{\text{int}} = \frac{(P_i)_{\text{coax}}}{(P_i)_{\text{isolated}}} = \left( \frac{2}{\sqrt{2}} \right)^{3/2} \sqrt{\frac{2}{2pA}} = \left( \frac{2}{\sqrt{2}} \right)^{3/2} \sqrt{\frac{2}{2pA}} = \sqrt{2} \quad (A6) $$

Also of interest is the net slipstream velocity, $w$, produced by the coaxial relative to two equivalent single rotor systems of the same radius and generating the same thrust. In the ideal case, the wake will contract to a value that is half the disk area, so from continuity considerations for the coaxial $w = 2(v_i)_c$. Because $(v_i)_c$ is a factor of $\sqrt{2}$ higher than found with the two single rotors under the present assumptions, the slipstream velocity of the coaxial is then a factor $2 \sqrt{2}/2 = 1.41$ greater that of the single rotor. While this is higher than that obtained under the assumptions for Cases 2 through 3, the result is still of some importance because the minimization of hover downwash and groundwash velocities from a rotor is important for several operational reasons. This means that to achieve the same (or similar values) of downwash velocities, a coaxial rotor needs to be operated at a lower effective disk loading than a single rotor machine when carrying the same weight.

**Case 2** – In practice, the two rotors of coaxial rotor system are never operated at the same thrust but instead at whatever individual thrust levels are necessary to achieve a balanced (equal and opposite) torque on the two rotors as a system. However, in the case where the two rotors are sufficiently close that they rotate in substantially the same plane at the same thrust, then they must also require the same torque (power). This is because both of the rotors share the same value of induced velocity (Fig. 2); it is, therefore, easily shown from Eq. A1 that

$$ P_u = T_u (v_i)_c \equiv T (v_i)_c = T \sqrt{\frac{2T}{2pA}} \quad (A7) $$

and

$$ P_l = T_l (v_i)_c \equiv T (v_i)_c = T \sqrt{\frac{2T}{2pA}} \quad (A8) $$

so that $P_u = P_l = P$. This means that for a coaxial rotor system with the rotors in the same plane operated at either the same thrust and/or the same torque then $\kappa_{\text{int}} = 1.414$. Notice that in the case where the rotors operate in the same plane then a torque balance can be achieved only if $T_u = T_l$; other variations of thrust sharing will spoil the torque balance.

**Case 3** – Generally, on practical coaxial designs the rotors are spaced sufficiently far apart that the lower rotor always operates in the *vena contracta* of the upper rotor. If it is assumed the lower rotor does not affect the wake convection of the upper rotor, then based on ideal flow considerations one-half of the disk area of the lower rotor must operate in the slipstream velocity induced by the upper rotor. This is a more difficult physical problem to model, in general, because it involves wake-blade interactions and local viscous effects, but it can initially be approached by a similar method of analysis using the principles of the simple momentum theory. However, the problem is further complicated by the fact that the rotors in the coaxial system can be operated at either equal thrust or at balanced torques.

The basic flow model for this case is shown in Fig. 2. In the first instance, assume that the two rotors operate at the same thrust, $T_u = T_l = T$. The induced velocity at the upper rotor is

$$ v_u = \sqrt{\frac{T}{2pA}} \quad (A9) $$

The *vena contracta* of the upper rotor is an area of $A/2$ with velocity $2v_u$. This represents the ideal case, although in practice the wake contraction may not be as large as this. Nevertheless, the ideal case represents the smallest fraction of the disk area affected by the upper rotor and so represents the minimum induced loss condition. Therefore, at the plane of the lower rotor there is a velocity of $2v_u + v_l$ over the inner one-half of the disk area – see Fig. 2.

Over the outer one-half of the disk area, the induced velocity is $v_l$. Assume that the velocity in the fully developed slipstream of the lower rotor (plane 3) is uniform with velocity $w_l$. The mass flow rate through the upper rotor is $\rho A v_u$, so that the momentum exiting in the slipstream of the upper rotor is $\rho A v_u (2v_u + v_l) = 2 \rho A v_u^2$. This is the momentum of the fluid into the lower rotor. The mass flow rates over the inner and outer parts of the lower rotor are $\rho (A/2) (2v_u + v_l)$ and $\rho (A/2) v_l$, respectively. Therefore,

$$ \dot{m} = \rho \frac{A}{2} (2v_u + v_l) + \rho \frac{A}{2} v_l = \rho A (v_u + v_l) \quad (A10) $$

The momentum flow out of plane 3 is $\dot{m} w_l$ assuming uniform velocity, so the thrust on the lower rotor is

$$ T_l = \rho A (v_u + v_l) w_l - 2 \rho A v_u^2 \quad (A11) $$
The work done by the lower rotor is

\[ P_l = T_l(v_u + v_l) \] (A12)

and this is equal to the gain in kinetic energy of the slipstream. Therefore

\[ T_l(v_u + v_l) = \frac{1}{2} \rho A (v_u + v_l) w_l^2 - \frac{1}{2} \rho \left( \frac{A}{2} \right) (2v_u)(2v_u)^2 \]

\[ = \frac{1}{2} \rho A (v_u + v_l) w_l^2 - 2 \rho A v_u^3 \] (A13)

Assuming \( T_l = T_u = T \), then \( T = 2 \rho A v_u^2 \), then from Eq. A11

\[ T_l = T = \frac{1}{2} \rho A (v_u + v_l) w_l \] (A14)

and from Eq. A13

\[ T(2v_u + v_l) = \frac{1}{2} \rho A (v_u + v_l) w_l^2 \] (A15)

Using Eqs. A14 and A15 gives \( w_l = 2v_u + v_l \) and substituting this into Eq. A14 and remembering that \( T = 2 \rho A v_u^2 \) gives

\[ 4 \rho A v_u^2 = \rho A (v_u + v_l) w_l = \rho A (v_u + v_l) (2v_u + v_l) \] (A16)

Rearranging as a quadratic in terms of \( v_l \) and solving gives

\[ v_l = \frac{-3 + \sqrt{77}}{2} v_u = 0.5616 v_u \] (A17)

The power for the upper rotor is \( P_u = T v_u = T v_h \) and for the lower rotor \( P_l = T(v_u + v_l) = 1.562 T v_h \). Therefore, for both rotors the total power is 2.562\( T v_h \). This is compared to 2\( T v_h \) when the rotors are operating in isolation. This means that in this case the induced power factor from interference, \( \kappa_{int} \), is given by

\[ \kappa_{int} = \frac{(P_l)_{coax}}{(2P_l)_{isolated}} = \frac{2.562 T v_h}{2 T v_h} = 1.281 \] (A18)

This case now represents a 28% increase in induced losses compared to a 41% increase when the two rotors have no vertical separation, i.e., \( \kappa_{int} = 1.281 \). This is closer to the values that can be indirectly deduced from most experiments – see, for example, Dingeldein (Ref. 19). Notice that based on the assumptions for Case 3, the net slipstream velocities for the coaxial will be increased to a value that is 28% higher than an equivalent single rotor when it is operated at the same disk loading.

**Case 4** – As previously alluded to, a comparison of performance on the basis of equal balanced torque between the upper and lower rotor is, however, a much more realistic operational assumption for a coaxial rotor. From the results from Case 3 (Eq. A11 and Eq. A13) then if there is now a torque balance \( P_u = P_l \) then

\[ T_u v_u = T_l (v_u + v_l) \] (A19)

Multiplying Eq. A11 by \( (v_u + v_l) \) and rearranging gives

\[ P_l (2v_u + v_l) = \rho A (v_u + v_l)^2 v_u w_l \] (A20)

Now, \( P_u = 2 \rho A v_u^3 = P_l \) and using Eq. A12 leads to

\[ P_l = \frac{1}{4} \rho A (v_u + v_l) w_l^2 \] (A21)

Substituting Eq. A21 into Eq. A20 and rearranging gives

\[ w_l = 4 v_u \left( \frac{v_u + v_l}{2v_u + v_l} \right) \] (A22)

Again, \( P_u = 2 \rho A v_u^3 = P_l \) and substituting Eq. A22 in Eq. A12 gives

\[ P_l = 2 \rho A v_u^3 = 8 \rho A (v_u + v_l) v_u^2 (v_u + v_l)^2 / (2v_u + v_l)^2 - 2 \rho A v_u^3 \] (A23)

which leads to

\[ 2v_l^3 + 5v_u v_l^2 + 2v_u^2 v_l - 2v_u^2 = 0 \] (A24)

Solving this cubic equation numerically leads to \( v_l = 0.4375 v_u \).

The induced power factor can now be evaluated. For the upper rotor \( (P/T)_u = v_h \) and for the lower rotor \( (P/T)_{int} = (v_u + v_l) \) so that for both rotors \( P/T = 2.4375 v_h \). This is compared to the value \( P/T = 2 v_h \) when the rotors are operating in isolation. This means that for coaxial rotor operation at balanced torque the induced power factor \( \kappa_{int} \) is reduced to

\[ \kappa_{int} = \frac{2.4375 v_h}{2 v_h} = 1.219 \] (A25)

This now represents a 22% increase compared to the case when the two rotors are operated in isolation, which is just slightly less than the 28% loss when the rotors are operated at equal thrusts. Notice that based on the assumptions for Case 4, the net average slipstream velocities for the coaxial will be 22% higher than an equivalent single rotor when operated at the same disk loading.